Engineering Notes

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Computation of Optimal Controls by Newton's Method Using a Discretized Jacobian

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Introduction

NEWTON'S method is a successive approximation technique to solve nonlinear two-point boundary value problems.[1,2] The nonlinear boundary value problem

$$\dot{x} = f(x,t) \tag{1}$$

$$\underset{m\times 1}{\underline{x}(t_0)} = \underline{x}_0,\quad \underset{(n-m)\times 1}{\bar{x}(t_f)} = \bar{x}_f \tag{2}$$

is solved by successively solving

$$\dot{x}^{(p+1)} = \frac{\partial f}{\partial x}(x^{(p)}(t),t)x^{(p+1)} + f(x^{(p)}(t),t) - \frac{\partial f}{\partial x}(x^{(p)}(t),t)x^{(p)} \tag{3}$$

with the boundary condition (2). The initial approximation $x^{(0)}(t)$ needs to be chosen judiciously for the process to converge. Convergence theorems under suitable hypotheses are given in Refs. 3 and 4.

Even for moderate-size problems, $\partial f/\partial x$ in Eq. (3) is a complicated expression. In the case of optimal control problems with bounds on control variables, it is even undefined for the resulting two-point boundary value problems.

In this Note suitable approximating expressions are found for the control variables, and the linear two-point boundary value problem given by Eqs. (3) and (2) is solved successively after discretizing the Jacobian. An advantage of our method is the introduction of no additional state variables. Here we consider optimal control problems with fixed terminal time and control-variable constraints.

The Jacobian $\partial f/\partial x$ can be discretized using

$$J(x^{(p)}(t),t) = (1/h)(f(x^{(p)}(t)+he_1,t) - f(x^{(p)}(t),t),\ldots,f(x^{(p)}(t)+he_n,t) - f(x^{(p)}(t),t)) \tag{4}$$

where $e_i = (0,\ldots,0,1,0,\ldots,0)^T$, 1 being in the ith place.

Solution of the Linear Problem

We need to solve successively for $p \geq 0$

$$\dot{x}^{(p+1)} = J(x^{(p)}(t),t)x^{(p+1)}(t) + f(x^{(p)}(t),t) - J(x^{(p)}(t),t)x^{(p)}(t) \tag{5}$$

with

$$\underline{x}^{(p+1)}(t_0) = \underline{x}_0, \bar{x}^{(p+1)}(t_f) = \bar{x}_f \tag{6}$$

Let

$$A(t) = J(x^{(p)}(t),t)$$
$$B(t) = f(x^{(p)}(t),t) - J(x^{(p)}(t),t)x^{(p)}(t) \tag{7}$$

Then Eq. (5) becomes

$$\dot{x}^{(p+1)} = A(t)x^{(p+1)}(t) + B(t) \tag{8}$$

Divide $[t_0,t_f]$ into N equally spaced intervals with $t_i = t_0 + i\Delta t, \Delta t$ being the length of each subinterval. Let

$$t_{i+\frac{1}{2}} = t_i + \frac{\Delta t}{2}, x^{(p)}_{i+\frac{1}{2}} = \frac{x^{(p)}(t_i) + x^{(p)}(t_{i+1})}{2} \tag{9}$$

and

$$\tilde{A}_{i+\frac{1}{2}} = J(x^{(p)}_{i+\frac{1}{2}}, t_{i+\frac{1}{2}}) \tag{10}$$

$$\tilde{B}_{i+\frac{1}{2}} = f(x^{(p)}_{i+\frac{1}{2}}, t_{i+\frac{1}{2}}) - J(x^{(p)}_{i+\frac{1}{2}}, t_{i+\frac{1}{2}})x^{(p)}_{i+\frac{1}{2}} \tag{11}$$

Equation (8) can be solved simply by

$$x^{(p+1)}(t_{i+1}) = \sum_{k=0}^{N} \tilde{A}^k_{i+\frac{1}{2}} x^{(p+1)}(t_i)\frac{\Delta t^k}{k!} + \sum_{k=0}^{M} \tilde{A}^k_{i+\frac{1}{2}} \tilde{B}_{i+\frac{1}{2}} \frac{\Delta t^{k+1}}{(k+1)!} \tag{12}$$

where $N \geq 2$, $M \geq 1$. In Ref. 5 it is demonstrated that Eq. (12) has a local truncation error of $O(\Delta t^3)$ under mild hypotheses. In our algorithms we use $N=2$, $M=1$.

We solve Eqs. (8) and (6) by finding the complete initial state $x^{(p+1)}(t_0)$. From Eq. (12) we can get a relation of the form

$$x^{(p+1)}(t_i) = C_i x^{(p+1)}(t_0) + D_i \tag{13}$$

The above equation can be normalized on the computer by dividing both sides by a suitable number so that C_i and D_i are not excessively large. In our algorithms the normalization is such that the maximal element of each C_i is 1.

From the relation

$$x^{(p+1)}(t_f) = C_N x^{(p+1)}(t_0) + D_N \tag{14}$$

we can solve for the unknown initial state $\bar{x}^{(p+1)}(t_0)$ using Eq. (6).

Approximations in Optimal Control Problems

Using Pontryagin's maximum principle,[6,7] an optimal control problem can be reduced to the solution of a nonlinear two-point boundary value problem. For problems with bounds on control variables, Newton's method cannot be applied directly since the resulting two-point boundary-value problem would have a discontinuous right-hand side. This means that $\partial f/\partial x$ is not even defined for certain values of the adjoint variables. We now indicate how to make certain approximations in this case.

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We consider bounds of the form

$$a_j \le u_j \le b_j, \qquad j=1,\dots,r \tag{15}$$

where a_j, b_j are constants and u_j is a control variable. Application of the maximum principle usually leads to a control law of the form

$$\begin{aligned} u&=a, & \eta&>0 \\ &=b, & \eta&<0 \end{aligned} \tag{16}$$

where η is a function of the state and adjoint variables. In Eq. (16) the subscripts are omitted for simplicity.

The signum function is given by

$$\begin{aligned} \mathrm{sgn}(\eta)&=+1, & \eta&>0 \\ &=-1, & \eta&<0 \end{aligned} \tag{17}$$

We consider the following approximations of Eq. (17):

$$s_1(t)=(2/\pi)\tan^{-1}\alpha\eta, \quad \alpha>0 \text{ is large} \tag{18}$$

$$\begin{aligned} s_2(t)&=+1, & \eta&>\frac{1}{k} \\ &=\eta k, & -\frac{1}{k}&\le\eta\le\frac{1}{k} \\ &=-1, & \eta&<\frac{-1}{k} \end{aligned} \tag{19}$$

where k is large.

Utilizing these, we can approximate Eq. (16) by

$$u=\frac{a+b}{2}+\frac{a-b}{2}\,\frac{2}{\pi}\tan^{-1}\alpha\eta \tag{20}$$

$$\begin{aligned} u&=a, & \eta&>\frac{1}{k} \\ &=\frac{a+b}{2}+\frac{a-b}{2}\eta k, & \frac{-1}{k}&\le\eta\le\frac{1}{k} \\ &=b, & \eta&<\frac{-1}{k} \end{aligned} \tag{21}$$

In the next section we compare Eqs. (20) and (21). Of course, several other approximations of Eq. (16) are possible.

A Numerical Example

All the computations for the example were performed on a VAX 11/730 in double precision.

We consider the orbital transfer problem treated in Ref. 1. The problem is to transfer a low-thrust ion rocket from the orbit of Earth to the orbit of Mars in fixed time with minimum fuel consumption. The orbits of Earth and Mars are assumed to be circular and coplanar, and the gravitational attractions of the two planets on the vehicle are neglected. The differential equations are given by[1]

$$\begin{aligned} \dot r&=w \\ \dot w&=\frac{v^2}{r}-\frac{K}{r^2}+\frac{c\beta}{m}\sin\theta \\ \dot v&=-\frac{wv}{r}+\frac{c\beta}{m}\cos\theta \\ \dot m&=-\beta \end{aligned} \tag{22}$$

where β and θ are the control variables. Here w and v are the radial and circumferential velocities, respectively; r is the radial distance from the sun, m the mass of the rocket, θ the thrust steering angle measured positive up from the local horizontal, K the gravitational constant, β the propellant mass flow, and c the constant equivalent exit velocity.

Letting $x_1=r$, $x_2=w$, $x_3=v$, and $x_4=m$, we get

$$\begin{aligned} \dot x_1&=x_2=f_1 \\ \dot x_2&=\frac{x_3^2}{x_1}-\frac{K}{x_1^2}+\frac{c\beta}{x_4}\sin\theta=f_2 \\ \dot x_3&=-\frac{x_2x_3}{x_1}+\frac{c\beta}{x_4}\cos\theta=f_3 \\ \dot x_4&=-\beta=f_4 \end{aligned} \tag{23}$$

The cost to be minimized is

$$P(\beta,\theta)=-x_4(t_f) \tag{24}$$

under the constraint

$$\beta_{\min}\le\beta\le\beta_{\max} \tag{25}$$

For this problem, after normalization,

$$\beta_{\min}=0,\ \beta_{\max}=0.075,\ c=1.872,\ K=1,\ t_0=0,\ t_f=3.816 \tag{26}$$

We apply Pontryagin's maximum principle[6,7] to get the conditions on the optimal control. The Hamiltonian is given by

$$H=\sum_{k=1}^{4}\psi_k f_k \tag{27}$$

The adjoint variables $\psi_1,\dots,\psi_4$ satisfy

$$\begin{aligned} \frac{d\psi_1}{dt}&=-\frac{\partial H}{\partial x_1}=\frac{\psi_2x_3^2}{x_1^2}-\frac{2K\psi_2}{x_1^3}-\frac{\psi_3x_2x_3}{x_1^2}=f_5 \\ \frac{d\psi_2}{dt}&=-\frac{\partial H}{\partial x_2}=-\psi_1+\frac{\psi_3x_3}{x_1}=f_6 \\ \frac{d\psi_3}{dt}&=-\frac{\partial H}{\partial x_3}=-\frac{2\psi_2x_3}{x_1}+\frac{\psi_3x_2}{x_1}=f_7 \\ \frac{d\psi_4}{dt}&=-\frac{\partial H}{\partial x_4}=\frac{\psi_2c\beta}{x_4^2}\sin\theta+\frac{\psi_3c\beta}{x_4^2}\cos\theta=f_8 \end{aligned} \tag{28}$$

Maximization of H with respect to θ and β yields

$$\sin\theta=\psi_2(\psi_2^2+\psi_3^2)^{-1/2},\quad \cos\theta=\psi_3(\psi_2^2+\psi_3^2)^{-1/2} \tag{29}$$

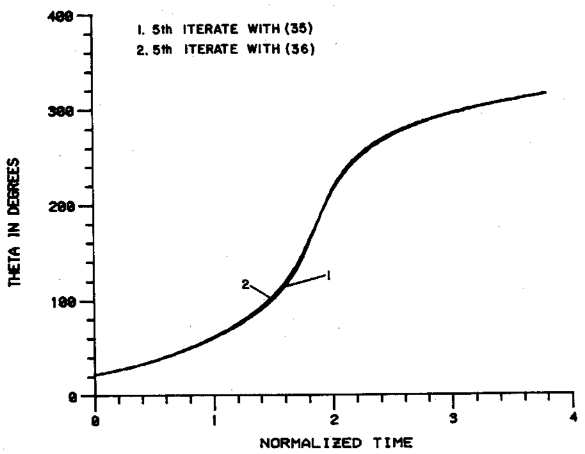

Fig. 1 Time history of control $\theta(t)$.

Table 1 Responses of numerical experiments 1 and 3

Experiment 1				
t	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$
0.000	0.1000000E+01	0.0000000E+00	0.1000001E+00	0.1000000E+01
0.191	0.1001306E+01	0.1557171E−01	0.1022928E+01	0.9858763E+00
0.382	0.1006640E+01	0.4207883E−01	0.1040806E+01	0.9718074E+00
0.572	0.1017978E+01	0.7820184E−01	0.1050970E+01	0.9578418E+00
0.763	0.1036940E+01	0.1214727E+00	0.1051064E+01	0.9441272E+00
0.954	0.1064491E+01	0.1666435E+00	0.1038516E+01	0.9318642E+00
1.145	0.1099163E+01	0.1939261E+00	0.1008555E+01	0.9288720E+00
1.336	0.1137883E+01	0.2106818E+00	0.9746130E+00	0.9280915E+00
1.526	0.1179118E+01	0.2204905E+00	0.9404853E+00	0.9276046E+00
1.717	0.1221648E+01	0.2244010E+00	0.9074050E+00	0.9272303E+00
1.908	0.1264422E+01	0.2231522E+00	0.8761121E+00	0.9269010E+00
2.099	0.1306520E+01	0.2174755E+00	0.8473410E+00	0.9265594E+00
2.290	0.1347196E+01	0.2084132E+00	0.8214525E+00	0.9261492E+00
2.480	0.1385881E+01	0.1966729E+00	0.7984498E+00	0.9255885E+00
2.671	0.1422087E+01	0.1823596E+00	0.7784108E+00	0.9246081E+00
2.862	0.1455043E+01	0.1598671E+00	0.7637146E+00	0.9199379E+00
3.053	0.1482061E+01	0.1229731E+00	0.7603067E+00	0.9072349E+00
3.244	0.1502035E+01	0.8679599E−01	0.7643012E+00	0.8935252E+00
3.434	0.1515375E+01	0.5362494E−01	0.7743755E+00	0.8655159E+00
3.625	0.1522751E+01	0.2442667E−01	0.7897511E+00	
3.816	0.1525000E+01	0.3526673E−05	0.8098027E+00	0.8513955E+00

Experiment 3				
t	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$
0.000	0.1000000E+01	0.0000000E+00	0.1000000E+01	0.1000000E+01
0.191	0.1001334E+01	0.1589859E−01	0.1023161E+01	0.9856900E+00
0.382	0.1006779E+01	0.4295490E−01	0.1041231E+01	0.9713800E+00
0.572	0.1018360E+01	0.7994082E−01	0.1051529E+01	0.9570700E+00
0.763	0.1037777E+01	0.1246559E+00	0.1051749E+01	0.9427600E+00
0.954	0.1065907E+01	0.1680761E+00	0.1036868E+01	0.9317997E+00
1.145	0.1100835E+01	0.1948448E+00	0.1006606E+01	0.9288802E+00
1.336	0.1139523E+01	0.2095615E+00	0.9724302E+00	0.9288802E+00
1.526	0.1180400E+01	0.2179346E+00	0.9387553E+00	0.9288802E+00
1.717	0.1222343E+01	0.2209009E+00	0.9065432E+00	0.9288802E+00
1.908	0.1264411E+01	0.2193964E+00	0.8763818E+00	0.9288802E+00
2.099	0.1305835E+01	0.2142730E+00	0.8485817E+00	0.9288802E+00
2.290	0.1345995E+01	0.2062634E+00	0.8232633E+00	0.9288802E+00
2.480	0.1384400E+01	0.1959754E+00	0.8004249E+00	0.9288802E+00
2.671	0.1420662E+01	0.1836125E+00	0.7800473E+00	0.9287337E+00
2.862	0.1453766E+01	0.1611001E+00	0.7649991E+00	0.9239540E+00
3.053	0.1481301E+01	0.1257015E+00	0.7602782E+00	0.9120646E+00
3.244	0.1501671E+01	0.8833577E−01	0.7642681E+00	0.8977546E+00
3.434	0.1515232E+01	0.5445575E−01	0.7743294E+00	0.8834446E+00
3.625	0.1522718E+01	0.2477358E−01	0.7897158E+00	0.8691346E+00
3.816	0.1524998E+01	0.2045698E−05	0.8098033E+00	0.8548246E+00

$$\beta=\beta_{\min} \qquad \eta\geq 0$$
$$=\beta_{\max} \qquad \eta<0 \tag{30}$$

where

$$\eta=\psi_4-(\psi_2^2+\psi_3^2)^{1/2}(c/x_4) \tag{31}$$

Let $x_i=\psi_{i-4}$, $i=5,\ldots,8$, $x=(x_1,\ldots,x_8)^T$, and $f=(f_1,\ldots,f_8)^T$. We now have a two-point boundary value problem of the form

$$\dot{x}=f(x) \tag{32}$$

with the normalized boundary conditions

$$\begin{aligned} x_1(0)&=1 & x_1(3.816)&=1.525\\ x_2(0)&=0 & x_2(3.816)&=0\\ x_3(0)&=1 & x_3(3.816)&=0.8098\\ x_4(0)&=1 \end{aligned} \tag{33}$$

and with

$$x_8(3.816)=\psi_4(3.816)=1 \tag{34}$$

which comes as a result of Eq. (24).

Since $\beta_{\min}=0$, $\beta_{\max}=.075$, we approximate Eq. (30) in the following manner using Eqs. (20) and (21):

$$\beta=.0375-.0375\left(\frac{2}{\pi}\right)\tan^{-1}(\alpha\eta) \tag{35}$$

$$\begin{aligned} \beta&=0, & \eta&\geq\frac{1}{k}\\ &=.0375-.0375\eta k, & -\frac{1}{k}&<\eta<\frac{1}{k}\\ &=.075, & \eta&\leq-\frac{1}{k} \end{aligned} \tag{36}$$

The constants $\alpha>0$ and $k>0$ are chosen to be as large as possible such that the numerical procedure still maintains stability.

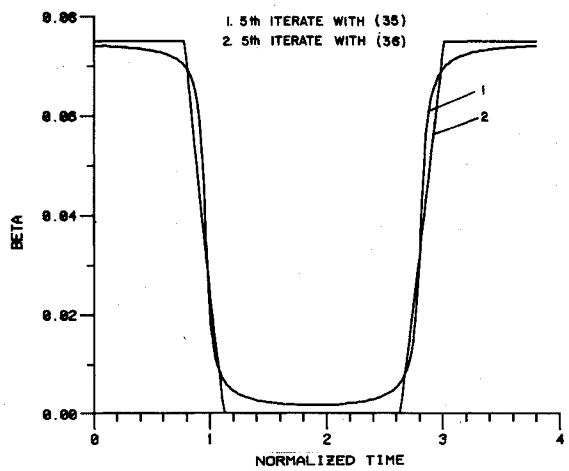

Fig. 2 Time history of control $\beta(t)$.

The Jacobian is discretized using Eq. (4) with $h=.000001$. We remark that h could be increased if needed. The initial function $x^{(0)}(t)$ is essentially the same as that in Ref. 1 except for $x_8(t)$, which is taken as 1 on the interval.

By using the approximations (35) and (36), we employ only 8 state variables, whereas the procedure in Ref. 1 employs 11 by converting the inequality constraint (25) into an equality constraint.

Let n denote the number of subintervals into which [0,3.186] is divided, i.e., $\Delta t=3.816/n$. We performed numerical experiments with the following parameters:

Experiment 1: $\alpha=20$ in Eq. (35), $n=200$

Experiment 2: $\alpha=20$ in Eq. (35), $n=40$

Experiment 3: $k=5$ in Eq. (36), $n=200$

Numerical experiments with $\alpha=25$ in Eq. (35) became unstable even when Δt is increased. On the other hand, with $\alpha=20$, increasing Δt did not affect the stability of the procedure as experiment 3 indicates. Increasing k to 7 in Eq. (36) made the procedure unstable with $n=200$. A better choice of $x^{(0)}(t)$ would of course increase the limiting values of α and k.

With experiment 1 the sequence $\{x^{(p)}\}$ converges to an accuracy of 4 significant digits in 5 iterations, and with experiment 2 a 4-digit accuracy was achieved at the seventh iterate. With experiment 3 the sequence converges to an accuracy of 4 significant digits in 5 iterations, and the fifth iterate has

$$\psi_1(0)=0.8794185E+00 \qquad \psi_2(0)=0.4203096E+00$$

$$\psi_3(0)=0.1027116E+01 \qquad \psi_4(0)=0.7651173E+00 \tag{37}$$

Figures 1 and 2 give plots of $\theta(t)$ vs t and $\beta(t)$ vs t for the fifth iterates of experiments 1 and 3. These compare very well with the exact plots given in Ref. 1. The values of θ and β from experiment 2 are close to those of experiment 1 and are not plotted. We note that the plots of β in Fig. 2 approximate the bang-bang control better than those given in Ref. 1.

To get an idea of the relative computing times involved, experiment 1 took 7 min and 30 s of execution time (VAX 11/730) to compute 5 iterates, whereas experiment 3 required 7 min and 50 s for 5 iterates.

Table 1 gives the output of experiments 1 and 3, and it can be observed that the responses are close.

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Near-Optimum Design of Large-Scale Systems

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Introduction

A PROPOSED scheme of decentralized near-optimization is applied to linear, time-invariant, interconnected large-scale systems. Due to the characteristics of such systems, the optimal control designs are, for the most part, necessarily near-optimum.[1-3] Several authors have applied different methods for near-optimum design of large-scale systems.[2] In the decentralized methods[1-2] the multilevel near-optimal controller design is based on minimizing the effect of interconnections, without reference to their possible beneficial roles. In this paper we propose a new scheme of decentralized near-optimum design based on using a multilevel feedback controller. It is assumed that the large-scale system is decomposed into ℓ subsystems, which are optimized by local feedback controllers with respect to a local performance index while ignoring the interactions among the subsystems. Then, the design of the global controller is performed using the available information on the subsystem level. To evaluate the near-optimization scheme, we compare it with that given by Siljak in Ref. 1, through applying both methods to a numerical example.

Near-Optimal Design

Consider the linear time-invariant interconnected system in the input-decentralized form[1]

$$\dot{x}_i=A_ix_i+b_iu_i+\sum_{\substack{j=1\\ j\neq i}}^{\ell}N_{ij}x_j, \quad i=1,\ldots,\ell \tag{1}$$

where $x_i \epsilon R^{n_i}$ and $u_i \epsilon R^{m_i}$. We assume that $x=0$ is the only equilibrium solution of Eq. (1).

In the proposed multilevel near-optimization scheme, the control $u_i(t)$ is considered as consisting of two parts, the local

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